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A New Lower Bound on the Minimal Length of a Binary Linear Code

M. C. Bhandari and M. S. Garg

In [4] Dodunekov and Manev have shown that $n(k, 2^{k-i}) \geqslant g(k, 2^{k-i}) + 2$ for $3 \leqslant i \leqslant k-4$. In case $k \geqslant 9$, we further improve this bound. The non-existence/existence of certain codes is established to prepare a table of bounds on $n(9, d)$ for $d \leqslant 2^8$.

1. Introduction

Let $n(k, d)$ be the smallest integer n for which there exists a binary $[n, k, d]$ = [length, dimension, minimum distance] code. In 1960, Griesmer [5] proved that

$$n(k, d) \geqslant \sum_{i=0}^{k-1} \lceil d/2^i \rceil \equiv g(k, d), \tag{1}$$

where $\lceil x \rceil$ denotes the smallest integer $\geqslant x$. This bound is called the Griesmer bound. In recent years much effort has been expanded on finding the exact value of $n(k, d)$ as a function of k and d, and determining value of k and d for which the Griesmer bound can be improved. In 1981, Helleseth [6] gave a characterization of all codes which meet the Griesmer bound for $d \leqslant 2^{k-1}$. In particular, his result gives the following.

Theorem 1.1 [6]. *For $2^{k-1} - 2^{k-i} + 3 \leqslant d \leqslant 2^{k-1} - 2^{k-i-1} - 2^i$, where $1 \leqslant i \leqslant \lfloor (k-2)/2 \rfloor$, the inequality $n(k, d) \geqslant g(k, d) + 1$ holds and for other values of $d \leqslant 2^{k-1}$, $n(k, d) = g(k, d)$.*

An improvement to this is given by the following theorem of Dodunekov and Manev [4].

Theorem 1.2 [4]. *For any $3 \leqslant i \leqslant k-4$, one has $n(k, 2^{k-i}) \geqslant g(k, 2^{k-i}) + 2$.*

In Section 2 of this paper we improve the lower bound given by Theorem 1.2. In Section 3 we use it, and other known results, to show the existence and non-existence of certain codes and for determining bounds on $n(9, d)$ for $d \leqslant 2^8$. The results are summarized by a table of bounds on $n(9, d)$.

If C is an $[n, k, d]$ code with $n = g(k, d) + t$, then C has a generator matrix in which every row has a weight between d and $d + t$ [4]. Moreover, if $d \leqslant 2^k$ and $n(k, d) \geqslant g(k, d) + t$, then $n(k+1, d) \geqslant g(k+1, d) + t$ [4]. A nice way of constructing codes of dimension $k-1$ is by considering residual code of k-dimensional codes. If C is a binary $[n, k, d]$ code, and if $c \in C$ and has a Hamming weight w ($\mathrm{wt}(c) = w$), then the code generated by the restriction of C to those columns in which c has zeros is called the residual code of C with respect to c (denoted by $\mathrm{res}(C, c)$ or by $\mathrm{res}(C, w)$). For

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$w < 2d$, $\text{res}(C, w)$ is an $[n-w, k-1, d_0]$ code with $d_0 \geqslant d - \lfloor w/2 \rfloor$. Thus, non-existence of an $[n-w, k-1, d_0]$ code will imply the non-existence of C. In [3], Dodunekov and Encheva have used this residual code technique to prove the following useful theorem.

THEOREM 1.3 [3]. *Let C be a binary $[n, k, d]$ code with $n = g(k, d) + t$, $t \leqslant 3$, $d = 2^{ms}$, $m \geqslant 2$ and $1 \leqslant s \leqslant m-1$. Suppose that $n(k-1, d/2) \geqslant g(k-1, d/2) + t$. If all weights in* $\text{res}(C, d)$ *are divisible by 2^s, then all weights in C are divisible by 2^{s+1}.*

An $[n, k, d]$ code C is called a maximal if there does not exist a proper supercode of C with the same n and d. If G is a generator matrix of a binary $[n, k, d]$ maximal code C and if there is an $x \in GF(2)^n$ the distance of which from C is m, then the matrix

$$\bar{G} = \left[\begin{array}{c|cccc} & \multicolumn{4}{c}{\leftarrow d-m \rightarrow} \\ x & 1 & 1 & \cdots & 1 \\ \hline G & \multicolumn{4}{c}{0} \end{array}\right]$$

generates an $[n+d-m, k+1, d]$ code. The following remark, which follows immediately from this observation, is useful in determining an upper bound for $n(k, d)$.

REMARK 1.4. If C is an $[n, k-1, d]$ maximal code of covering radius R, then $n(k, d) \leqslant n + d - R$.

Since the covering radius of the $[128, 8, 64]$ first order Reed–Muller code is 56 [9], by Remark 1.4 we have the following:

THEOREM 1.5. $n(9, 64) \leqslant 136$.

For given k and d, let $b(k, d) = n(k+1, d) - n(k, d)$. In case $b(k, d) = 1$, the following theorem shows the existence of codes of covering radius $d-1$.

THEOREM 1.6 [1]. *The covering radius R of an $[n(k, d), k, d]$ code satisfies $R \leqslant d - b(k, d)$. Moreover, if $b(k, d) = 1$, then there exists an $[n(k, d), k, d]$ code of covering radius $d-1$.*

The MacWilliams identities are helpful in showing the non-existence of certain codes. They are given by the following theorem.

THEOREM 1.7 [8, *p.* 127]. *Let C be a binary linear code and let $C^\perp$ be its dual code. Let $\{A_i\}$ and $\{B_i\}$, $0 \leqslant i \leqslant n$, be the weight distributions of C and $C^\perp$ respectively. Then*

$$|C|\, B_m = \sum_{i=0}^{n} A_i K_m(i), \qquad 0 \leqslant m \leqslant n,$$

where

$$K_m(x) = \sum_{j=0}^{m} (-1)^j \binom{n-x}{m-j} \binom{x}{j}, \qquad 0 \leqslant m \leqslant n,$$

are the Krawtchouk polynomials.

In the rest of this paper, $\{A_i\}$ and $\{B_i\}$, $0 \leqslant i \leqslant n$, will denote the weight distribution of a code C and its dual $C^\perp$ respectively.

Since $n(k, 2d) = n(k, 2d-1) + 1$, throughout this paper, unless otherwise specified, d is assumed to be even.

2. New Lower Bounds on $n(k, d)$

In case $k \geqslant 9$, the following theorem improves the lower bound on $n(k, d)$ given by Theorem 1.2.

THEOREM 2.1. *If* $3 \leqslant i \leqslant k-4$, $k \geqslant 9$, *then* $n(k, 2^{k-i}) \geqslant g(k, 2^{k-i}) + 3$.

PROOF. We prove the theorem by induction, first on k for $i = 3$ and then on i. Let $i = 3$. If $k = 9$ then, by Theorem 1.2, $n(9, 64) \geqslant g(9, 64) + 2 = 131$. If C is a $[131, 9, 64]$ code then $\mathrm{res}(C, 64)$ will be a $[67, 8, 32]$ code which does not exist as $n(8, 32) = 68$ [10]. Hence $n(9, 64) \geqslant 132 = 9(9, 64) + 3$. If $k > 9$ and if C is a $[g(k, 2^{k-3}) + 2, k, 2^{k-3}]$ code, then $\mathrm{res}(C, 2^{k-3})$ is a $[g(k-1, 2^{k-4}) = 2, k-1, 2^{k-4}]$ code, a contradiction to the induction hypothesis. Suppose that $i > 3$ and that the statement is true for $i - 1$. If $k > 9$, then $n(k-1, 2^{k-i}) = n(k-1, 2^{k-1-(i-1)}) \geqslant g(k-1, \; 2^{k-1-(i-1)}) = 3$ and hence $n(k, 2^{k-i}) \geqslant g(k, 2^{k-i}) + 3$. If $k = 9$ then $i = 4$ or 5, and for each of these Verhoeff's table [10] gives the required bounds. □

Another lower bound for $n(k, d)$ for certain values of d is given by the following theorem.

THEOREM 2.2. *If* $d = 2^{k-4} - 2^m$, $k \geqslant 10$, $m \geqslant 0$, *then* $n(k, d) \geqslant g(k, d) + 3$.

For the proof we need the following lemma which has been shown by Ivanov [7]; however, we sketch an independent proof which also demonstrates some of the techniques used by us in the construction of the table on $n(9, d)$ given at the end of this paper.

LEMMA 2.3. $n(10, 60) \geqslant 126$.

PROOF. If $n(9, 60) \geqslant 125$, then $n(10, 60) \geqslant 126$. Otherwise, $n(9, 60) = 124$ [10]. Let C be a binary $[124, 9, 60]$ code. Note that $124 = g(9, 60) + 2$ and $\mathrm{res}(C, 60)$ is a $[64, 8, 30]$ code. Since $64 = g(8, 30) + 2$, $\mathrm{res}(C, 60)$ has a generator matrix G in which each row is of weight 30, 31 or 32. If a row of G has weight 31 then $\mathrm{res}(\mathrm{res}(C, 60), 31)$ is a $[33, 7, 15]$ code. But a code with these parameters does not exist [10]. Hence by Theorem 1.3 all weight in C must be divisible by 4. Using the technique of residual codes and Table I in [10], it is easy to verify that the possible non-zero weights in C are 60, 64, 72, 76, 88, 104, 120 and 124. Moreover, $B_2 = 0$. For, if $B_2 \neq 0$, then by elementary row and column operations any generator matrix for C can be put in the form

$$\left[\begin{array}{cc|c} 1 & 1 & * \; * \; * \cdots\cdots * \\ \cdots & \cdots & \cdots\cdots\cdots\cdots \\ & 0 & G' \end{array}\right].$$

But then G' generates a $[122, 8, 60]$ code which does not exist [10]. The MacWilliams identities for B_0, B_1 and B_2 are

$$A_{60} + A_{64} + A_{72} + A_{76} + A_{88} + A_{104} + A_{120} + A_{124} = 511, \tag{2}$$

$$A_{60} + A_{64} + 5A_{72} + 7A_{76} + 13A_{88} + 21A_{104} + 29A_{104} + 29A_{120} + 31A_{124} = 31, \tag{3}$$

$$27A_{60} + 27A_{64} - 69A_{72} - 165A_{76} - 645A_{88} - 1733A_{104} - 3333A_{120} - 3813A_{124} = 3813. \tag{4}$$

If $A_{124} \neq 0$, then $A_{124} = 1$ and $A_{72} = A_{76} = A_{88} = A_{104} = A_{120} = 0$ (otherwise, the sum of any two such codewords gives a codeword of weight < 60). But then, by (3), $A_{60} = 0$. Hence $A_{124} = 0$. Similarly, $A_{120} = 0$. If $A_{104} \neq 0$, then $A_{104} = 1$ and $A_{88} = 0$. A linear combination of (2) and (4) gives $96A_{72} + 192A_{76} = 8224$. This is not possible, as $3 \nmid 8224$. Hence $A_{104} = 0$. If both A_{88} And A_{76} are zero, then solving (2), (3) and (4) we obtain $A_{64} = -41$, a contradiction. Hence either $A_{88} \neq 0$ or $A_{88} = 0$ and $A_{76} \neq 0$. If $A_{88} \neq 0$, then $C_1 = \mathrm{res}(C, 88)$ is a $[36, 8, 16]$ code for which $b(8, 16) \geq 2$ [10]. Hence, by Theorem 1.6, $R(C_1) \leq 14$. Permuting columns if necessary, a generator matrix for C can be put in the form

$$\left[\begin{array}{cccc|ccc} & \longleftarrow 88 \rightarrow & & & & & \\ 1 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ \hline & & A & & & G_1 & \end{array}\right],$$

where G_1 is a generator matrix for C_1. So $R(C) \leq \lfloor 88/2 \rfloor + R(C_1) \leq 58$ [1, Theorem 1]. Similarly, if $A_{76} \neq 0$, then also $R(C) \leq 58$. Thus the value of $b(9, 60)$ must be greater than 1: for, if $b(9, 60) = 1$ then, by Theorem 1.6, there is a $[124, 9, 60]$ code of covering radius 59. Hence $n(10, 60) = n(9, 60) + b(9, 60) \geq 126$. □

Proof of Theorem 2.2. We prove the theorem by induction on m. If $k = 10$, then $0 \leq m \leq 5$. If $m = 0$, 1, 4 or 5, then the result follows by consideration of residual codes, and bounds on $n(9, d)$ from [10, 11]. If $m = 2$ or 3, then $d = 60$ or 56, and the result follows using Lemma 2.3 or [3]. If $k > 10$ and $m = 0$ then, by Theorem 2.1, $n(k, 2^{k-4} - 1) \geq g(k, 2^{k-4} - 1) + 3$. Suppose that $d = 2^{k-4} - 2^m$, $m > 0$, $k > 10$ and that the statement is true for $m - 1$. Let C be a $[g(k, d) + 2, k, d]$ code. Then $\mathrm{res}(C, d)$ is a $[g(k - 1, d/2) + 2, k - 1, d/2]$ code which, by assumption, does not exist. □

We show the non-existence of certain codes in the following three theorems.

Theorem 2.4. $n(9, 96) \geq 196$.

Proof. Suppose a $[195, 9, 96]$ code C exists. Since $195 = g(9, 96) + 2$, there exists a generator matrix G for C in which each row has weight 96, 97 or 98. $A_{97} = 0$: for, if $A_{97} \neq 0$, then $\mathrm{res}(C, 97)$ is a $[98, 8, 48]$ code which does not exist [10, 11]. So all weights in C are even. Using techniques of residual codes and Table I in [10] it is easy to verify that 96, 144, 192 and 194 are the possible non-zero weights in C. The MacWilliams identities for B_0 and B_1 are

$$A_{96} + A_{144} + A_{192} + A_{194} = 511, \tag{4}$$

$$-3A_{96} + 93A_{144} + 189A_{192} + 193A_{194} = 195. \tag{5}$$

$A_{194} = 0$: for, if $A_{194} \neq 0$, then $A_{194} = 1$ and $A_{144} = A_{192} = 0$ and hence, by (5), $A_{96} = -2/3$, a contradiction. If $A_{192} = 0$ then $A_{192} = 1$ and $A_{144} = 0$, and hence, by (5), $A_{96} < 0$. So $A_{192} = 0$, and on solving (4) and (5) simultaneously we obtain $A_{96} = 493$ and $A_{144} = 18$. Let $c_1, c_2 \in C$ with $\mathrm{wt}(c_1) = \mathrm{wt}(c_2) = 144$. Without loss of generality, we can assume that c_1 and c_2 have the following configuration:

$$\begin{array}{l} 1\,1 \cdot\cdot\cdot\cdot\cdot\; 1\;\; 1\,0\,0\,0 \cdot\cdot\cdot\; 0\;\; 0 \quad c_1 \\ 1\,1 \cdot\cdot 1\,0\,0 \cdots 0\,1\,1 \cdot\cdot 1\,0\,0 \cdots 0 \quad c_2 \\ \qquad\qquad\quad \leftarrow 48 \rightarrow \end{array}$$

Then res(C, c_1) is a $[51, 8, 24]$ code having a vector of weight 48. This is not possible, as a $[51, 8, 24]$ code has the unique weight distribution $A_0 = 1$, $A_{24} = 204$ and $A_{32} = 51$. So C does not exist. □

THEOREM 2.5. $n(9, 112) \geqslant 228$.

PROOF. Suppose that a $[227, 9, 12]$ code exists. Proceeding as in the proof of Lemma 2.3, it is easy to verify that all weights in res(C, d) are even and hence, by Theorem 1.3, all weights in C are divisible by 4. Using the techniques of residual codes and Table I in [10, 11], it is easy to see that 112, 128, 176 and 124 are the possible non-zero weights in C. The MacWilliams identities for B_0, B_1 and B_2 are

$$A_{112} + A_{128} + A_{176} + A_{224} = 511, \tag{6}$$

$$-3A_{112} + 29A_{128} + 125A_{176} + 221A_{224} = 227, \tag{7}$$

$$-109A_{112} + 307A_{128} + 7699A_{176} + 24\,307A_{224} = -25\,651 + 512B_2. \tag{8}$$

If $A_{224} \neq 0$, then $A_{224} = 1$ and $A_{120} = A_{176} = 0$. But then (6) and (7) are inconsistent. So $A_{224} = 0$. Similarly, if $A_{176} \neq 0$ then $A_{176} = 1$ and on solving (6), (7) and (8) we obtain $B_2 = -2$, a contradiction. Therefore $A_{176} = 0$. Solving (6), (7) and (8) we obtain $B_2 = -14$, a contradiction. □

THEOREM 2.6. $n(9, 176) \geqslant 355$.

PROOF. If possible, let C be a $[354, 9, 176]$ code. Proceeding as in the proof of Lemma 2.3, it is easy to see that 176, 192, 224, 228 and 352 are the possible non-zero weights in C. If $A_{352} \neq 0$, let $c_1, c_2 \in C$ such that wt$(c_1) = 352$ and wt$(c_2) = 176$. Then by permuting coordinates, if necessary we can assume that c_1 and c_2 have the following configuration

$$\begin{array}{l} 1\ 1\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ 1\ 0\ 0 \quad c_1, \\ 1\ 1\ \cdot\ \cdot\ 1\ 0\ 0\ \cdot\ \cdot\ 0\ 0\ 0 \quad c_2. \end{array}$$

For, if the last two coordinates of c_2 are 10, 01 or 11, then wt$(c_1 + c_2) = 178$. Since C has a generator matrix in which every row has weight 176, deleting the last two coordinates we obtain a $[225, 9, 112]$ code. This contradicts Theorem 2.5. Hence $A_{352} = 0$. If $A_{228} \geqslant 2$, let c_1, $c_2 \in C$, wt$(c_1) = $ wt$(c_2) = 228$. Permuting coordinates, if necessary it can be assumed that they have the following configuration

$$\begin{array}{l} 1\ 1\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \quad \cdot\ \ 1\ 0\ 0\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ 0 \quad c_1, \\ 1\ 1\ \cdot\ \cdot\ \cdot\ 1\ 0\ 0\ \cdots\ 0\ 1\ 1\ \cdot\ \cdot\ \cdot\ 1\ 0\ \cdot\ \cdot\ 0 \quad c_2. \\ \qquad\quad \longleftarrow x \longrightarrow \ \longleftarrow x \longrightarrow \end{array}$$

where $x = 88$, 96, 112 or 114. Then res(C, c_1) has a vector of weight x and hence res(res$(C, c_1), x)$ is a $[126 - x, 7, 62 - \lfloor x/2 \rfloor]$, code, which does not exist [10]. So $A_{228} \leqslant 1$. The MacWilliams identities for B_0, B_1 and B_2 are

$$A_{176} + A_{192} + A_{224} + A_{228} = 511, \tag{9}$$

$$-A_{176} + 15A_{192} + 47A_{224} + 51A_{22}8 = 177, \tag{10}$$

$$175A_{176} - 273A_{192} - 4241A_{224} - 5025A_{228} = 62\,481 - 512B_2. \tag{11}$$

If $A_{228} = 1$ then $A_{224} = 0$ and, by (9) and (10), $A_{192} = 636/16$, a contradiction. Hence $A_{228} = 0$. Similarly, $A_{224} = 0$. On solving (9), (10) and (11) we obtain $B_2 = -15$, a contradiction. □

3. Bounds on $n(9, d)$

By Theorem 1.1, $n(9, d) = g(9, d)$ for all $d \leqslant 256$ except for $3 \leqslant d \leqslant 126$, $131 \leqslant d \leqslant 188$ and $195 \leqslant d \leqslant 216$. For each of these values of d, $n(9, d) \geqslant g(9, d) + 1$. If $d \leqslant 58$, Table I of Verhoeff [10] gives bounds on $n(9, d)$. Dodunekov and Encheva have further improved these lower bounds [3] for $d = 24$, 28, 30 and 56. Lower bounds for other values of d can be obtained by making use of results established in Section 2 or by showing the non-existence of certain codes by the residual code technique.

An upper bound on $n(9, d)$ for $d \geqslant 60$ is determined by one of the following methods.

3.1. Use Remark 1.4. This requires a lower bound on the covering radius of an $[n, 8, d]$ maximal code. Concatenation is a known way of constructing new codes. For example, if G_1 is a generator matrix of the $[128, 8, 64]$ first order Reed–Muller code C_1 and if G_2 is a generator matrix of an $[n, 7, d]$ maximal code C_2, then the code C generated by the matrix

$$G = \left[G_1 \,\middle|\, \begin{matrix} 0 & 0 & \cdots & 0 \\ & G_2 & & \end{matrix} \right]$$

is the concatenation of C_1 and C_2. If the first row of G_1 has weight 128 and $d \leqslant 164$, then C is an $[128 + h, d + 64]$ code and $R(C) \geqslant R(C_1) + R(C_2) = 56 + R(C_2)$ [2]. Hence, by Remark 1.4,

$$n(9, d + 64) \leqslant 128 + n + d + 64 - 56 - R(C_2) = 136 + n + d + 64 - R(C_2).$$

If $d \in S = \{i \mid 1 \leqslant i \leqslant 8 \text{ or } 21 \leqslant i \leqslant 24 \text{ or } i = 15, 16\}$ then $n(8, d) - n(7, d) = 1$ [10]. Hence, for each $d \in S$, by Theorem 1.6 there exists an $[n(7, d), 7, d]$ code of covering radius $d - 1$. On replacing C_2 by each such code, we have the following.

Theorem 3.1. *Let S be as defined above. Then, for each $d \in S$, $n(9, d + 64) \leqslant 137 + n(7, d)$.*

On putting the value for $n(7, d)$ for each even $d \in S$ we have the following.

Corollary 3.2. $n(9, 66) \leqslant 145$, $n(9, 68) \leqslant 149$, $n(9, 70) \leqslant 153$, $n(9, 72) \leqslant 156$, $n(9, 80) \leqslant 172$, $n(9, 86) \leqslant 184$ and $n(9, 88) \leqslant 187$.

3.2. Use of bound given by Dodunekov and Manev [4]. Dodunekov and Manev have shown that if $2^{k-1} - 2^{k-i} + 3 \leqslant d \leqslant 2^{k-1} - 2^{k-i-1} - 2^i$, $2 \leqslant i \leqslant [(k-2)/2]$, then $n(k, d) \leqslant 2^k - 2^{k-v} + n(k - v, d - 2^{k-1} + 2^{k-v-1})$, for all $1 \leqslant v \leqslant i - 1$. This is used for determining an upper bound for $132 \leqslant d \leqslant 188$ and $196 \leqslant d \leqslant 216$.

3.3. Constructing certain nine-dimensional codes. Let C_1 be the $[136, 9, 64]$ code constructed in the proof of Theorem 1.5 and let G_1 be a generator matrix for C_1 the first row of which is of weight 128. If G_2 is a generator matrix for an $[n, 8, d]$ code C_2 with $d \leqslant 64$, then the matrix

$$G = \left[G_1 \,\middle|\, \begin{matrix} 0 & 0 & \cdots & 0 \\ & G_2 & & \end{matrix} \right]$$

generates a $[136 + n, 9, 64 + d]$ code. Thus if n_1 is the upper bound on $n(8, d)$ from [10, 11], then $n(9, d) \leqslant n_1 + 136$ for $d \leqslant 64$.

TABLE 1

d	$q(9, d)$	$n(9, d)$	d	$g(9, d)$	$n(9, d)$
60	122	124–132	138	280	281–282
62	126	129–134	140	283	284–285
64	129	132–136	142	287	288–290
66	137	138–145	144	290	291–292
68	140	141–148	146	296	297–298
70	144	145–153	148	299	300–301
72	147	148–153	150	303	304
74	152	153–166	152	306	307
76	155	156–168	154	311	312–314
78	159	161–170	156	314	315–318
80	162	164–172	158	318	319–321
82	168	169–180	160	321	322–324
84	171	172–182	162	328	329–331
86	175	177–184	164	331	332–334
88	178	180–187	166	335	336–337
90	183	185–204	168	338	339–341
92	186	188–206	170	343	344–346
94	190	192–208	172	346	347–349
96	193	196–210	174	350	351–352
98	200	202–212	176	353	355
100	203	205–214	178	359	360–362
102	207	209–217	180	362	363–365
104	210	212–220	182	366	367–368
106	215	217–226	184	369	370–371
108	218	220–229	186	374	375–377
110	222	224–232	188	377	378–380
112	225	228–235	196	395	396
114	231	233–242	198	399	400
116	234	236–244	200	402	403
118	238	240–246	202	407	408
120	241	243–248	204	410	411
122	246	248–250	206	414	415–416
124	249	251–252	208	417	418–419
126	253	254	210	423	424
132	268	269	212	426	427
134	272	273	214	430	431
136	275	276	216	433	434

3.4. Deleting coordinates from an $[n, 9, d']$ code with $d' > d$. If G is a generator matrix for an $[n, k, d']$ code with first row of weight d', then on deleting any i ($<d'$) columns which have a non-zero entry in the first row, one obtains an $[n-i, k, d'-i]$ code. For example, if C is the $[256, 9, 128]$ first order Reed–Muller code, then on deleting suitable i ($i \leqslant 14$) coordinates we obtain a $[256-i, 9, 128-i]$ code. Therefore, $n(9, 128-i) \leqslant 256-i$ for $i \leqslant 14$.

We conclude by compiling, in Table 1, the bounds on $n(9, d)$ for all even d's, $60 \leqslant d \leqslant 216$, for which $n(9, d) > g(9, d)$.

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